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Zeros of Networked Systems with Time-invariant Interconnections

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Abstract

This paper studies zeros of networked linear systems with time-invariant interconnection topology. While the characterization of zeros is given for both heterogeneous and homogeneous networks, homogeneous networks are explored in greater detail. In the current paper, for homogeneous networks with time-invariant interconnection dynamics, it is illustrated how the zeros of each individual agent's system description and zeros definable from the interconnection dynamics contribute to generating zeros of the whole network. We also demonstrate how zeros of networked systems and those of their associated blocked versions are related.

Key words: Networked Systems. Multi-agent systems. Zeros

1 Introduction

Recent developments of enabling technologies such as communication systems, cheap computation equipment and sensor platforms have given great impetus to the creation of networked systems. Thus, this area has attracted significant attention worldwide and researchers have studied networked systems from different perspectives (see e.g. [33], [28], [35]). In particular, in view of the recent chain of events [15], [10] and [30], the issues of security and cyber threats to the networked systems have gained growing attention. This paper uses system theoretic approaches to deal with problems involved with the security of networks.

Recent works have shown that control theory can be used as an effective tool to detect and mitigate the effects of cyber attacks on the networked systems; see for example [25], [6], [17], [1], [34], [36] and the references listed therein. The authors of [36] have introduced the concept of *zero-dynamics attacks* and shown how attackers can use knowledge of networks' zeros to produce control commands such that they are not detected as security threats [1]. They have further shown that zeros of networks provide valuable information relevant to detecting cyber attacks. The authors in [36] were more concerned with mitigating such attacks and did not provide a detailed discussion about zeros of the networked systems. In addition to this, even though various aspects of the dynamics of networked systems have been extensively studied in the literature [29,27,11], to the authors' best knowledge, the zeros of networked systems have not been studied in any detail except in [44]. The current paper establishes a link between the problem of zero-dynamics attacks and the analysis of zeros that has been recorded in [44]. Furthermore, several new results are introduced in the current paper compared to its preliminary conference version including the provision of proofs of certain results which were not part of the conference version.

This paper examines the zeros of networked systems in more depth. Our focus is on networks of finite-dimensional linear discrete-time dynamical systems that arise through static interconnections of a finite number of such systems. Such models arise naturally in applications of linear networked systems, e.g. for cyclic pursuit [24]; shortening flows in image processing [5], or for the

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[1] This is discussed further in the next section.

discretization of partial differential equations [4].

Our ultimate goal is to analyze the zeros of networked systems with periodic, or more generally time-varying interconnection topology. An important tool for this analysis is blocking or lifting technique for networks with time-invariant interconnections. Note that blocking of linear time-invariant systems is useful if not standard in design of controllers for linear periodic systems as shown by [7] and [23]. The authors of [3] and [16] have examined zeros of blocked systems obtained from blocking of time-invariant systems. Their works have been extended in [43] and [8]. However, these earlier contributions do not take any underlying network structure into consideration. In this paper, we introduce some results that provide a first step in that direction.

It is worthwhile noting the blocking technique has been used in the networked systems literature for both control and identification purposes. For instance, the authors in [19] have exploited this technique to identify the system parameters in a networked system via the subspace approach. The same set of authors have employed the blocking technique to study moving horizon estimation problem for networked systems [18]. In [26] the authors have utilized the blocking technique to provide a sufficient and necessary condition for stability of a class of networked systems with communication bandwidth limitation. A similar problem has been addressed in [14] using the blocking.

The structure of this paper is as follows. First, in Section 2 we introduce state-space and higher order polynomial system models for time-invariant networks of linear systems. A central result used is the strict system equivalence between these different system representations. Moreover, we completely characterize both finite and infinite zeros of arbitrary heterogeneous networks. For homogeneous networks of identical SISO systems more explicit results are provided in Section 3. Homogeneous networks with a circulant coupling topology are studied as well. In Section 4, a relation between the transfer function of the blocked system and the transfer function of the associated unblocked system is explained. We then relate the zeros of blocked networked systems to those of the corresponding unblocked systems, generalizing work in [43], [8], [42]. Finally, Section 5 provides the concluding remarks.

2 Problem Statement and Preliminaries

We consider networks of N linear systems, coupled through constant interconnection parameters. Each agent is assumed to have the state-space representation as a linear discrete-time system

$$\begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i v_i(t) \\ w_i(t) &= C_i x_i(t),\ i = 1, \dots, N. \end{aligned} \tag{1}$$

Here, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{p_i \times n_i}$ are the associated system matrices. We assume that each system is reachable and observable and that the agents are interconnected by static coupling laws

$$v_i(t) = \sum_{j=1}^{N} L_{ij} w_j(t) + R_i u(t) \in \mathbb{R}^{m_i} \tag{2}$$

with $L_{ij} \in \mathbb{R}^{m_i \times p_j}$, $R_i \in \mathbb{R}^{m_i \times m}$ and $u(t) \in \mathbb{R}^m$ denoting an external input applied to the whole network. Further, we assume that there is a p-dimensional interconnected output given by

$$y(t) = \sum_{i=1}^{N} S_i w_i(t) + Du(t) \quad \text{with} \quad S_i \in \mathbb{R}^{p \times p_i},\ i = 1, \dots, N. \tag{3}$$

Define $\overline{m} = \sum_{i=1}^N m_i$, $\overline{p} = \sum_{i=1}^N p_i$, $\overline{n} = \sum_{i=1}^N n_i$ and *coupling matrices*

$$L = (L_{ij})_{ij} \in \mathbb{R}^{\overline{m} \times \overline{p}} \quad R = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix} \in \mathbb{R}^{\overline{m} \times m}$$
$$S = (S_1, \dots, S_N) \in \mathbb{R}^{p \times \overline{p}} \quad D \in \mathbb{R}^{p \times m}$$

as well as *node matrices*

$$\begin{aligned} &A = \operatorname{diag}(A_1, \dots, A_N), \quad B = \operatorname{diag}(B_1, \dots, B_N) \\ &C = \operatorname{diag}(C_1, \dots, C_N), \quad x(t) := \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} \in \mathbb{R}^{\overline{n}}. \end{aligned} \tag{4}$$

Then the closed-loop system is

$$\begin{aligned} x(t+1) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) + Du(t), \end{aligned} \tag{5}$$

with matrices

$$\mathbf{A} := A + BLC \quad \mathbf{B} := BR, \quad \mathbf{C} := SC. \tag{6}$$

One could also start by assuming that each system (1) is defined in terms of a restricted version of Rosenbrock-type equations [31] i.e. by systems of higher order difference equations

$$\begin{aligned} T_i(\sigma)\xi_i(t) &= U_i(\sigma) v_i(t) \\ w_i(t) &= V_i(\sigma)\xi_i(t). \end{aligned} \tag{7}$$

Here σ denotes the shift operator that acts on sequences of vectors $(\xi(t))_t$ as $(\sigma\xi(t)) = \xi(t+1)$. Furthermore, T_i, U_i, V_i denote polynomial matrices of sizes $T_i(z) \in \mathbb{R}[z]^{r_i \times r_i}, U_i(z) \in \mathbb{R}[z]^{r_i \times m_i}$ and $V_i(z) \in \mathbb{R}[z]^{p_i \times r_i}$, respectively. We always assume that $T_i(z)$ is *nonsingular*, i.e. that $\det T_i(z)$ is not the zero polynomial. Moreover, the system (7) is assumed to be strictly proper, i.e. we assume that the associated transfer function

$$G_i(z) = V_i(z)T_i(z)^{-1}U_i(z) \tag{8}$$

is strictly proper. Following Fuhrmann [12], any strictly proper system of higher order difference equations has an associated state-space realization (A, B, C), the so-called **shift realization**, such that the polynomial matrices

$$\begin{pmatrix} zI - A & -B \\ C & 0 \end{pmatrix}, \quad \begin{pmatrix} T(z) & -U(z) \\ V(z) & 0 \end{pmatrix} \tag{9}$$

are strict system equivalent [12]. If the first order representation (1) is strict system equivalent to the higher order system (7) then of course the associated transfer functions coincide, i.e. we have

$$C_i(zI - A_i)^{-1}B_i = V_i(z)T_i(z)^{-1}U_i(z). \tag{10}$$

Throughout this paper we assume that the first order and higher order representations i.e. the systems (1) and (7), are chosen to be of minimal order, respectively. This is equivalent to the controllability and observability of the shift realizations (1) associated with these representations (7). It is also equivalent to the simultaneous left coprimeness of $T_i(z), U_i(z)$ and the right coprimeness of $T_i(z), V_i(z)$. Proceeding as above, define polynomial matrices

$$T(z) = \operatorname{diag}(T_1(z), \ldots, T_N(z)) \in \mathbb{R}[z]^{\overline{r} \times \overline{r}} \tag{11}$$

and similarly for $V(z)$ and $U(z)$. Here $\overline{r} = \sum_{i=1}^{N} r_i$. Using this notation, we write all N systems of (7) in the matrix form as

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w(t) = \begin{pmatrix} T(\sigma) & -U(\sigma) \\ V(\sigma) & 0 \end{pmatrix} \begin{pmatrix} \xi(t) \\ v(t) \end{pmatrix}, \tag{12}$$

where $w(t) = \begin{pmatrix} w_1(t)^\top & w_2(t)^\top & \ldots & w_N(t)^\top \end{pmatrix}^\top$ and similarly for $\xi(t)$ and $v(t)$. Then we have the left- and right coprime factorizations of the $\overline{p} \times \overline{m}$ *node transfer function* as

$$G(z) = C(zI - A)^{-1}B = V(z)T(z)^{-1}U(z).$$

The interconnections are given, as before, by

$$\begin{aligned} v(t) &= Lw(t) + Ru(t) \\ y(t) &= Sw(t) + Du(t). \end{aligned}$$

The resulting network representation then becomes

$$\begin{pmatrix} 0 \\ I \end{pmatrix} y(t) = \begin{pmatrix} T(\sigma) - U(\sigma)LV(\sigma) & -U(\sigma)R \\ SV(\sigma) & D \end{pmatrix} \begin{pmatrix} \xi(t) \\ u(t) \end{pmatrix} \tag{13}$$

with the $p \times m$ *network transfer function* defined as

$$\begin{aligned} \Gamma(z) &= \mathbf{C}(zI - \mathbf{A})^{-1}\mathbf{B} + D \\ &= SV(z)(T(z) - U(z)LV(z))^{-1}U(z)R + D. \end{aligned} \tag{14}$$

2.1 Zero-dynamics Attacks

In the previous subsection, we introduced the formulation for the networked systems. This subsection deals with the notion of the zero-dynamics attacks that can be considered as one of the motivations for studying zeros of the networked systems.

Let us assume that the input signal $u(t)$ is contaminated by a false signal say $u^a(t) \in \mathbb{R}^m$. This signal might be designed by one or more intruders to lead the system (5) into an unsafe region. Hence, under an attack the command signal applied to the system (5) is no more $u(t)$ but is $\bar{u}(t) = u(t) + u^a(t)$. It is apparent that when $u^a(t)$ is equal to zero for all t the system is operating in its nominal condition. We write the dynamics of the system (5) under an attack as

$$\begin{aligned} \bar{x}(t+1) &= \mathbf{A}\bar{x}(t) + \mathbf{B}u(t) + \mathbf{B}u^a(t) \\ \bar{y}(t) &= \mathbf{C}\bar{x}(t) + Du(t) + Du^a(t), \end{aligned} \tag{15}$$

where $\bar{x}(t) \in \mathbb{R}^{\bar{n}}, \bar{y}(t) \in \mathbb{R}^p$.

It is practical to assume that there exists an anomaly detector, say $\mathcal{A}_{\mathcal{D}}(\bar{u}(t), \bar{y}(t))$, that makes decision about the operational status of the system (15) using the information from $\bar{u}(t)$ and $\bar{y}(t)$. The anomaly detector delivers a residue, say $\gamma(t) \in \mathbb{R}^d$. This residue may activate the alarm system if $||\gamma(t)|| \geq \epsilon$ for some $\epsilon > 0$.

We now exploit the superposition theorem, and write the state and output vectors of the system (15) as $\bar{x}(t) = x(t) + x^a(t)$ and $\bar{y}(t) = y(t) + y^a(t)$ with

$$\begin{aligned} x^a(t+1) &= \mathbf{A}x^a(t) + \mathbf{B}u^a(t) \\ y^a(t) &= \mathbf{C}x^a(t) + Du^a(t). \end{aligned} \tag{16}$$

Without loss of generality, in the computation of the vector $\bar{x}(t)$ we assume that $x^a(0) = x(0)$. This implies that in calculation of $\bar{x}(t)$, one can consider that $x(t)$ only contains the component corresponding to the input $u(t)$ and $x^a(t)$ includes terms due to the initial condition $x^a(0) = x(0)$ and the attack signal $u^a(t)$.

Futhermore, note that the vector $y^a(t)$ can be revealed by a fault detection filter see e.g. [37]. Here, we naturally assume that $||\gamma(t)|| := ||y^a(t)||$.

Definition 1 *Consider the system* (16) *with* $x^a(t_0) = 0$. *The attack sequence* $\{u^a(t_0), \ldots, u^a(t_k)\}$ *with* $u^a(t_i) \neq 0$ *is said to be undetectable if* $y^a(t) = 0$, $\forall t > t_0$.

The undetectable attacks have the form [21]

$$u^a(t) = u^a(0)z_0^t,$$

where $z_0 \in \mathbb{C}$ is a *zero* of the networked system defined by the quadruple $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, D\}$ and $u^a(0) \neq 0$.

It is easy to see that for a value $z_0 \in \mathbb{C}$ that is not a pole of $\Gamma(z)$ and appropriate initial conditions, $y^a(t) = \Gamma(z_0)u^a(0)z_0^t = 0$ $\forall t$ if z_0 is a zero of $\Gamma(z)$ [21]. Thus, the nonzero attack signal $u^a(t)$ defined above remains undetectable if z_0 is a zero of $\Gamma(z)$. It is worthwhile noting that here we exclude zeros at the origin as they are associated with attack signals with zero energy.

As just shown, the undetectable attacks are very closely related to zeros [2] of the networked system (5). Thus, in the subsequent parts of this paper, we will provide a detailed study about zeros of the system (5). First, the next subsection formally defines zeros for the system (5).

2.2 Zeros of Networked Systems

We first state the following general definition.

Definition 2 *Consider the proper transfer function matrix* $\bar{G}(z) = \bar{W}(z) + \bar{V}(z)\bar{T}(z)^{-1}\bar{U}(z)$ *where* $\bar{T}(z) \in \mathbb{R}[z]^{\tilde{r}\times\tilde{r}}$, $\bar{W}(z)$, $\bar{V}(z)$, $\bar{U}(z)$ *are polynomial matrices in a minimal Rosenbruck-type realization of* $\bar{G}(z)$.

A ***finite zero*** *of the polynomial system matrix*

$$\bar{\Pi}(z) = \begin{pmatrix} \bar{T}(z) & -\bar{U}(z) \\ \bar{V}(z) & \bar{W}(z) \end{pmatrix} \tag{17}$$

is any complex number $z_0 \in \mathbb{C}$ *such that*

$$\text{rank } \bar{\Pi}(z_0) < \text{grk } \bar{\Pi}(z)$$

holds. $\bar{\Pi}(z)$ *is said to have a* ***zero at infinity*** *if*

$$\tilde{r} + \text{rank } \lim_{z\to\infty} \bar{G}(z) < \text{grk } \bar{\Pi}(z).$$

We now introduce the following definition that is an immediate consequence of Definition 2 and standard in systems and control literature [22].

Definition 3 *Consider the proper transfer function matrix* $\bar{G}(z)$ *with minimal realization* $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ *and* $\bar{A} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$. *Then Then, finite zeros of* $\bar{G}(z) = \bar{D} + \bar{C}(zI - \bar{A})^{-1}\bar{B}$ *are defined to be the finite values of* z *for which the rank of the following system matrix falls below its normal rank*

$$\bar{\Sigma}(z) = \begin{bmatrix} zI - \bar{A} & -\bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}.$$

Further, $\bar{G}(z)$ *is said to have an infinite zero when* $\tilde{n} + \text{rank}\,\bar{D}$ *is less than the normal rank of* $\bar{\Sigma}(z)$, *or equivalently the rank of* $\bar{D}$ *is less than the normal rank of* $\bar{G}(z)$.

Note that the **normal rank** grk $\bar{G}(z)$ of a rational matrix function $\bar{G}(z)$ is defined as

$$\text{grk } \bar{G}(z) = \max\{\text{rank } \bar{G}(z) \mid z \in \mathbb{C}, \bar{G}(z) \neq \infty\}.$$

The zeros defined in the above definition capture a class of zeros known as *invariant zeros* in the literature. The following remark comments on different existing notions of zeros studied in the systems and control literature.

Remark 1 *Those zeros defined through the system matrix are referred to as* ***invariant zeros*** *in the literature [31]. There also exists the notion of* ***transmission zeros*** *which are obtainable from the Smith-McMillan form of a transfer function matrix. It is worthwhile noting that when a realization is minimal, invariant zeros and transmission zeros coincide. However, when it is not minimal the invariant zeros include the transmission zeros. Furthermore, all unreachable and unobservable modes known as* ***input decoupling zeros*** *and* ***output decoupling zeros*** *comprise the remaining zeros.*

Additionally, we assert the following lemma from [13] that helps us to relate zeros of $\bar{\Pi}(z)$ and $\bar{\Sigma}(z)$.

Lemma 1 *Let* $\bar{G}(z) = \bar{W}(z) + \bar{V}(z)\bar{T}(z)^{-1}\bar{U}(z) = \bar{D} + \bar{C}(zI - \bar{A})^{-1}\bar{B}$ *be left and right coprime factorizations of the proper transfer function* $\bar{G}(z)$, *respectively. For*

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[2] This is why this set of attacks are known as zero-dynamics attacks.

each $\tilde{q} \geq \max(\tilde{n}, \tilde{r})$ there exist unimodular polynomial matrices $E(z), F(z)$ such that

$$E(z)\begin{pmatrix} I_{\tilde{q}-\tilde{n}} & 0 & 0 \\ 0 & zI - \bar{A} & -\bar{B} \\ 0 & \bar{C} & \bar{D} \end{pmatrix} F(z) = \begin{pmatrix} I_{\tilde{q}-\tilde{r}} & 0 & 0 \\ 0 & \bar{T}(z) & -\bar{U}(z) \\ 0 & \bar{V}(z) & \bar{W}(z) \end{pmatrix}.$$

From the results of Lemma 1, we express the following corollary.

Corollary 1 *Consider the interconnection $L.R, S, D$ associated with the system* (5)*. Then the polynomial system matrix*

$$\Pi(z) = \begin{pmatrix} T(z) - U(z)LV(z) & -U(z)R \\ SV(z) & D \end{pmatrix} \qquad (18)$$

has a finite or infinite zero if and only if the system matrix

$$\Sigma(z) = \begin{pmatrix} zI - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & D \end{pmatrix}$$

of the associated shift realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ has a finite or infinite zero.

This leads to a complete characterization of the zeros for the interconnected system (5) as explained in the subsequent theorem. We emphasize that the characterization of the zeros in the subsequent Theorem 1 holds for any interconnection matrices and does not require any assumptions on reachability or observability of the network, except of those for the individual node systems.

Theorem 1 ([13]) *Consider the strictly proper node transfer function $G(z)$ with minimal representations (4) as*

$$G(z) = C(zI - A)^{-1}B = V(z)T(z)^{-1}U(z).$$

Let L, R, S, D be any arbitrary constant interconnection matrices of the proper dimensions and let $\Gamma(z)$ as defined in (14) denote the orresponding network transfer function. Assume that $G(z)$ is represented by a polynomial left coprime matrix fraction description (MFD) as

$$G(z) = D_L^{-1}(z)N_L(z).$$

Then

(1) For all $z \in \mathbb{C}$

$$\operatorname{rank}\begin{pmatrix} zI - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & D \end{pmatrix} = \overline{n} - \overline{r} + \operatorname{rank}\begin{pmatrix} T(z) - U(z)LV(z) & -U(z)R \\ SV(z) & D \end{pmatrix}.$$

(2) For all $z \in \mathbb{C}$

$$\operatorname{rank}\begin{pmatrix} zI - A - BLC & -BR \\ SC & D \end{pmatrix} = \overline{n} - \overline{p} + \operatorname{rank}\begin{pmatrix} D_L(z) - N_L(z)L & -N_L(z)R \\ S & D \end{pmatrix}.$$

(3) $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ has a finite zero at $z_0 \in \mathbb{C}$ if and only if

$$\operatorname{rank}\begin{pmatrix} T(z_0) - U(z_0)LV(z_0) & -U(z_0)R \\ SV(z_0) & D \end{pmatrix} < \overline{r} + \operatorname{grk}\ \Gamma(z).$$

(4) $(A + BLC, BR, SC, D)$ has a zero at infinity if and only if

$$\operatorname{rank}\ D < \operatorname{grk}\ \Gamma(z).$$

In particular, if D has full-row rank or full-column rank, then $(A + BLC, BR, SC, D)$ has no infinite zero.

3 Zeros of Homogeneous Networks

The preceding result has a nice simplification in the case of **homogeneous networks of SISO agents**, i.e. where the node systems (A_i, B_i, C_i) are single input single output systems with identical transfer function. Let us define the *interconnection transfer function* as

$$\phi(z) = S(zI - L)^{-1}R + D.$$

The next theorem relates the zeros of the system (5) to those of the interconnection dynamics [3] defined by the quadruple (L, R, S, D). Before we provide this main result, we need to state the following lemma regarding the generic rank of $\Gamma(z)$.

[3] The term interconnection dynamics is partly a misnomer. There is no dynamics separate to that included within the agent description, and the interconnecting matrices are all constant. The transfer function $\phi(z)$ is a theoretical construct: it is the transfer function from $u(t)$ to $y(t)$ resulting when every system is replaced by z^{-1}.

Lemma 2 *Assume that (A_i, b_i, c_i) are scalar SISO systems with identical transfer function $g(z) = c_i(zI - A_i)^{-1}b_i$. Let L, R, S, D denote any constant interconnection matrices of the proper dimensions and $\phi(z) = S(zI - L)^{-1}R + D$ be the interconnection transfer function. Then with $\Gamma(z)$ as defined in (14), the following equality holds.*

$$\operatorname{grk}\ \Gamma(z) = \operatorname{grk}\ \phi(z).$$

Proof. Consider any coprime factorization $g(z) = \frac{p(z)}{q(z)}$ of the strictly proper transfer function $g(z)$, having McMillan degree n. Define $h(z) = g(z)^{-1} = \frac{q(z)}{p(z)}$. First observe that

$$\begin{pmatrix} zI - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ \mathbf{C}(zI - \mathbf{A})^{-1} & I \end{pmatrix} \begin{pmatrix} zI - \mathbf{A} & -\mathbf{B} \\ 0 & \Gamma(z) \end{pmatrix}.$$

Then we can easily write

$$\operatorname{grk}\ \Gamma(z) = \operatorname{grk} \begin{pmatrix} zI - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & D \end{pmatrix} - nN. \tag{19}$$

Then by applying the second part of Theorem 1, one obtains

$$\begin{aligned}
&\operatorname{grk} \begin{pmatrix} zI - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & D \end{pmatrix} = \\
&N(n-1) + \operatorname{grk} \begin{pmatrix} q(z)I_N - p(z)L & -p(z)R \\ S & D \end{pmatrix} = \\
&N(n-1) + \operatorname{grk} \begin{pmatrix} h(z)I_N - L & -R \\ S & D \end{pmatrix} = \\
&N(n-1) + \operatorname{grk} \begin{pmatrix} \eta I_N - L & -R \\ S & D \end{pmatrix} = \\
&Nn + \operatorname{grk}\ \phi(z).
\end{aligned} \tag{20}$$

By substituting the last equality of (20) into (19), the result follows.

□

Theorem 2 *Assume that (A_i, b_i, c_i) are SISO systems with identical transfer function $g(z) = c_i(zI - A_i)^{-1}b_i$. Then $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ has a zero at infinity if and only if (L, R, S, D) has a zero at infinity.*

Proof. By Lemma 2, the network transfer function matrix $\Gamma(z)$ and the interconnection transfer matrix $\phi(z)$ have the same normal rank. Using the conclusion of Theorem 1 (part 4), the result follows. □

Theorem 2 shows that the infinite zero structure of a homogeneous network depends only upon the interconnection parameters and not on the specific details of the node transfer function. This is in contrast to the finite zero structure, as is shown by the following result.

Theorem 3 *Assume that (A_i, b_i, c_i) are SISO systems with identical transfer function $g(z) = c_i(zI - A_i)^{-1}b_i$. Let $p(z)/q(z)$ be a coprime polynomial factorization of $g(z)$ and define $h(z) = g(z)^{-1}$. Let (L, R, S, D) denote any constant interconnection matrices of the proper dimensions.*

(1) $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ *has a finite zero at $z_0 \in \mathbb{C}$ with $p(z_0) \neq 0$ if and only if $h(z_0) \in \mathbb{C}$ is a finite zero of (L, R, S, D).*
(2) $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ *has a finite zero at $z_0 \in \mathbb{C}$ with $p(z_0) = 0$ if and only if (L, R, S, D) has a zero at infinity.*

Proof. We first prove the first part of the theorem. By Lemma 2 and Theorem 1, $z_0 \in \mathbb{C}$ is a zero of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ if and only if

$$\operatorname{rank} \begin{pmatrix} q(z_0)I_N - p(z_0)L & -p(z_0)R \\ S & D \end{pmatrix} < N + \operatorname{grk}\ \phi(z). \tag{21}$$

For $p(z_0) \neq 0$ this is equivalent to

$$\operatorname{rank} \begin{pmatrix} h(z_0)I_N - L & -R \\ S & D \end{pmatrix} < N + \operatorname{grk}\ \phi(z),$$

i.e. $h(z_0)$ being a finite zero of (L, R, S, D). For the second part note that $z_0 \in \mathbb{C}$ is a zero of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ if and only if inequality (21) holds. If $p(z_0) = 0$, then by coprimeness of $p(z)$ and $q(z)$ we have $q(z_0) \neq 0$ and therefore (21) is equivalent to

$$N + \operatorname{rank} D = \operatorname{rank} \begin{pmatrix} q(z_0)I_N & 0 \\ S & D \end{pmatrix} < N + \operatorname{grk}\ \phi(z).$$

This is equivalent to $\operatorname{rank}\ D < \operatorname{grk}\ \phi(z)$. Thus a zero of the node transfer function $g(z)$ is a zero of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ if and only if (L, R, S, D) has a zero at infinity. This completes the proof. □

Now assume that D has full-column rank or full-row rank. Then the homogeneous network realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ has no zeros at infinity. Thus in this case the finite zeros of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ are exactly the preimages of the finite zeros of (L, R, S, D) under the rational function $h(z)$. We conclude with a result that is useful for the design of networks with prescribed zero properties.

The result below bears a certain similarity with a result by Fax and Murray[11]. As shown by them, a formation of N identical vehicles can be analyzed for stability by analyzing a single vehicle with the same dynamics modified by only a scalar, which assumes values equal to the eigenvalues of the interconnection matrix. Such a result is to do with poles, linking those of the individual agent and the overall system via the eigenvalues (which are pole-like) of the interconnection matrix. Our result is to do with the zeros, but still links those of the individual agent, those of the interconnection matrix (suitably interpreted) and those of the whole system.

With the help of the preceding results, we can now study two other important properties of networks, namely, losslessness and passivity. It is well known, see e.g. [40] (Section II. B), that if all agent transfer function matrices and the system defined by the quadruple (L, R, S, D) are lossless, then the system (5) is lossless. We now provide an improvement of this result for the case of SISO agents.

Recall that a strictly proper real rational transfer function $g(z)$ is called lossless [39] if all poles of $g(z)$ are in the open unit disc and $|g(z)| = 1$ holds for all $|z| = 1$. A key property used below is that $|g(z)| > 1$ if $|z| < 1$ and $|g(z)| < 1$ if $|z| > 1$.

Theorem 4 *Assume that D has full-column rank or full-row rank. Then*

(1) The homogeneous network $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ has no zeros at infinity. A complex number z_0 is a finite zero of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ if and only if $h(z_0) \neq \infty$ is a finite zero of (L, R, S, D).
*(2) Assume that the agent transfer function $g(z)$ is **lossless**. Then $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ is a minimum phase network, i.e. all of its zeros have absolute value < 1, if and only if (L, R, S, D) is minimum phase.*

Proof. The first claim is an immediate consequence of Theorem 3. If $g(z)$ is lossless then $|g(z)| < 1$ holds if and and only if $|z| > 1$. Thus $h(z) = 1/g(z)$ maps the complement of the open unit disc onto itself. Thus $|z| \geq 1$ if and only if $|h(z)| \geq 1$. Therefore (L, R, S, D) has a finite zero η_0 with $|\eta_0| \geq 1$ if and only if each z with $h(z) = \eta_0$ satisfies $|z| \geq 1$ and is a zero of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$. Note that for any finite η_0, there is necessarily a z satisfying $h(z) = \eta_0$, since this is a polynomial equation for z. This proves the result. □

We now extend the second part of the above corollary for the choice of **passive** transfer functions [39]. Let us recall that $g(z)$ is passive if and only if

(1) all poles of $g(z)$ are in $|z| \leq 1$
(2) $|g(z)| \leq 1 \;\; \forall \;\; |z| = 1$.

This implies

(1) $|g(z)| < 1 \;\; \forall \, |z| > 1$
(2) If $|g(z)| > 1$, then $|z| < 1$.

Corollary 2 *Assume that D has full-column rank or full-row rank and $g(z)$ is **passive**. Then $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ is a minimum phase network, i.e. all of its zeros have absolute value < 1, if (L, R, S, D) is minimum phase.*

Proof. Suppose $|z_0|$ is a finite zero of $\{A, B, C, D\}$. Then $h(z_0)$ is a finite zero of (L, S, R, D), i.e., $1/g(z_0)$ is a finite zero of (L, S, R, D). By the minimum phase assumption, $|1/g(z_0)| < 1$ or $|g(z_0)| > 1$. Passivity of $g(z_0)$ thus implies $|z_0| < 1$. □

It is worthwhile that passive and loss-less transfer functions can be considered as classes of inner functions that is discussed in the literature [32].

At this stage, we comment on the position of zeros with respect to the unit circle. Suppose that z_0 is a zero of the system (5). Then it is said to be a minimum phase zero if $|z_0| < 1$ and non-minimum phase zero if $|z_0| \geq 1$.

We make the following observation that is inspired by Theorem 7 in [38].

Suppose that the state matrix $\mathbf{A}$ is Schur stable [20]. Then even though those attacks that exploit minimum phase zeros remain undetectable, they give rise to states that asymptotically converge to the origin. On the other hand, those attacks associated with non-minimum phase zeros are undetectable and the states corresponding to such attack signals are unbounded as $t \to \infty$.

3.1 Design of Networks

In this subsection, we study construction of networks that exhibits no finite zeros. We derive a simple sufficient condition for homogeneous networks. By Theorem 4, the homogeneous network $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ is zero-free if and only if (L, R, S, D) is zero-free. For simplicity, we assume that there is a single external input and a single external output associated with the network, i.e. $m = p = 1$. Moreover, we assume $D = 0$. Thus the interconnection transfer function $\phi(z) = S(zI_N - L)^{-1}R$ is scalar strictly proper rational. The next result characterizes which outputs of the SISO interconnected system lead to a network without finite zeros, for given state and input interconnection matrices.

Theorem 5 (SISO Design Condition) *Assume that (A_i, b_i, c_i) are identical minimal SISO systems with identical transfer functions. Let (L, R) be reachable with $L \in \mathbb{R}^{N \times N}, R \in \mathbb{R}^N$. Then a network output $S \in \mathbb{R}^{1 \times N}$ defines a minimal network realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ without finite zeros if and only if $S(zI_N - L)^{-1}R$ has relative degree N.*

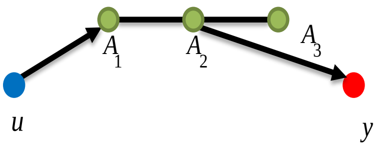

Fig. 1. *A homogenous network consisting of three SISO agents. The agents, the external input and measurement are depicted by green, blue and red circles, accordingly. The whole network has two zeros at -1 and 1 when all weights are set to unity.*

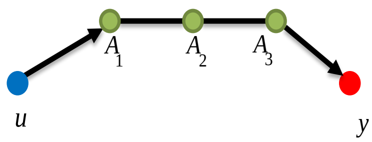

Fig. 2. *A homogenous network consisting of three SISO agents. The agents, the external input and measurement are depicted by green, blue and red circles, accordingly. The whole network is zero-free when all weights are set to unity.*

Proof. By Theorem 4, the homogeneous network $(\mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ has no finite zeros if and only if this holds for $(L, R, S, 0)$. In the SISO case this is equivalent to the transfer function $S(zI_N - L)^{-1}R$ having no zeros. By [13], $(\mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ is minimal if and only if (L, R, S) is minimal. In either case, $S(zI_N - L)^{-1}R$ has McMillan degree N and has no zeros if and only if the relative degree of $S(zI_N - L)^{-1}R$ is equal to N. $\square$

The above theorem characterizes when the SISO networked systems are zero-free. We note that the condition is equivalent to the sytem-theoretic condition that the closed loop system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, 0)$ is feedback irreducible; i.e. that $(\mathbf{A} + \mathbf{B}K, \mathbf{C})$ is observable for all state feedback matrices K.

The next example illustrates that the zeros of the system (5) may drastically change by replacing and adding a link.

Example 1 *Consider the network depicted in Fig. 1 where the nodes are simply double integrators. Note that there exist bidirectional links between the agents. By assuming a unit weight on each link, it is easy to verify that for such a network the interconnection matrices are* $L = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. *Moreover, the interconnection dynamics has a single zero at* $z = 1$. *Hence, by using Theorem 3 it is easy to see that the whole network has two zeros at* 1 *and* -1. *One can also observe that by adding an extra link in Fig. 1 from agent* A_3 *to the measurement node, with the same set of interconnection matrices as before except for* S *which assumes random values in its nonzero entry, the whole network becomes zero-free. The same result holds i.e. the resultant network is zero-free, when the topology is modified according to Fig. 2.*

3.2 Circulant Homogeneous Networks

Homogeneous networks with special coupling structures appear in many applications, such as cyclic pursuit [24]; shortening flows in image processing [5] or the discretization of partial differential equations [4]. Here, we characterize the zeros for interconnections that have a circulant structure. A homogeneous network is called **circulant** if the state-to-state coupling matrix L is a circulant matrix, i.e.

$$L = \mathrm{Circ}(\mathrm{c}_0, ..., \mathrm{c}_{N-1}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & \cdots & c_{N-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{pmatrix}.$$

The book [9] provides algebraic background on the circulant matrices. A basic fact on circulant matrices is that they are simultaneously diagonalizable by the **Fourier matrix**

$$\Phi = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2N-2} \\ \vdots & & & & \\ 1 & \omega^{N-1} & \omega^{2N-2} & \dots & \omega^{(N-1)^2} \end{pmatrix},$$

where $\omega = e^{2\pi j/N}$ denotes a primitive $N-$th root of unity. Note, that Φ is both a unitary and a symmetric matrix. It is then easily seen that any circulant matrix L has the form $L = \Phi \mathrm{diag}\,(p_L(1), p_L(\omega), \dots, p_L(\omega^{N-1}))\Phi^*$, where $p_L(z) := \sum_{k=0}^{N-1} c_k z^{k-1}$. As a consequence of the preceding analysis we obtain the following result.

Theorem 6 *Suppose that the system in (5) is a circulant homogeneous network. Let D be full rank and $M = \mathrm{diag}\,(p_L(1), \dots, p_L(\omega^{N-1}))$ and $w_1, \dots, w_N$ denote the complex roots of*

$$\det \begin{pmatrix} wI_N - M & -\Phi^* R \\ S\Phi & D \end{pmatrix} = 0.$$

Then

$$\bigcup_{k=1}^{N} \{z \in \mathbb{C} \mid q(z) - w_k p(z) = 0\}$$

are the finite zeros of the homogeneous network $(\mathbf{A},\mathbf{B},\mathbf{C},D)$.

Proof. By Theorem 3, we conclude that the system defined by $(\mathbf{A},\mathbf{B},\mathbf{C},D)$ has a finite nonzero zero if and only if the following matrix pencil has less than full rank

$$\begin{pmatrix} h(z)I_N - L & -R \\ S & D \end{pmatrix}. \tag{22}$$

Observe that the following equality holds

$$\begin{gathered}\begin{pmatrix} h(z)I_N - M & -\Phi^* R \\ S\Phi & D \end{pmatrix} = \\ \begin{pmatrix} \Phi^* & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} h(z)I_N - L & -R \\ S & D \end{pmatrix}\begin{pmatrix} \Phi & 0 \\ 0 & I \end{pmatrix}.\end{gathered} \tag{23}$$

Note that multiplication of a matrix by non-singular matrices on the left and right respectively does not change the rank. This implies the result. □

4 Zeros of Blocked Networked Systems

The technique of blocking or lifting a signal is well-known in systems and control [7] and signal processing [39]. In systems theory, this method has been mostly exploited to transform linear discrete-time periodic systems into linear time-invariant systems in order to apply the well-developed tools for linear time-invariant systems; see [2] and the literature therein. As we mentioned in the introduction section, the blocking technique has been used in the networked systems literature for both identification and control purposes see e.g. [19], [14]. Here, we demonstrate how blocking can be applied to the networked system (5). Furthermore, we show how the system matrix and zeros of the resultant blocked system relate to the those of the corresponding unblocked system.

Let us consider the following networked system

$$\begin{aligned} x(t+1) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) + Du(t), \end{aligned} \tag{24}$$

with matrices

$$\mathbf{A} := A + BLC \quad \mathbf{B} := BR, \quad \mathbf{C} := SC.$$

and the network transfer function

$$\Gamma(z) = D + SC(zI - A - BLC)^{-1}BR.$$

Here $x(t) \in \mathbb{R}^{\overline{n}}$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ and $A = \operatorname{diag}(A_1,\dots,A_N), B = \operatorname{diag}(B_1,\dots,B_N), C = \operatorname{diag}(C_1,\dots,C_N)$ are block-diagonal. Given an integer $T \geq 1$ as the block size, we define for $t = 0, T, 2T, \dots$

$$\begin{aligned} U(t) &= \begin{pmatrix} u(t)^\top & u(t+1)^\top & \dots & u(t+T-1)^\top \end{pmatrix}^\top, \\ Y(t) &= \begin{pmatrix} y(t)^\top & y(t+1)^\top & \dots & y(t+T-1)^\top \end{pmatrix}^\top. \end{aligned}$$

The **blocked system** then is defined as [2]

$$\begin{aligned} x(t+T) &= \mathbf{A}_b x(t) + \mathbf{B}_b U(t) \\ Y(t) &= \mathbf{C}_b x(t) + \mathbf{D}_b U(t), \end{aligned} \tag{25}$$

where

$$\mathbf{A}_b = \mathbf{A}^T, \quad \mathbf{B}_b = \begin{pmatrix} \mathbf{A}^{T-1}\mathbf{B} & \mathbf{A}^{T-2}\mathbf{B} & \dots & \mathbf{B} \end{pmatrix},$$

$$\mathbf{C}_b = \begin{pmatrix} \mathbf{C}^\top & \mathbf{A}^\top\mathbf{C}^\top & \dots & \mathbf{A}^{(T-1)^\top}\mathbf{C}^\top \end{pmatrix}^\top,$$

$$\mathbf{D}_b = \begin{pmatrix} D & 0 & \dots & 0 \\ \mathbf{CB} & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{CA}^{T-2}\mathbf{B} & \mathbf{CA}^{T-3}\mathbf{B} & \dots & D \end{pmatrix}. \tag{26}$$

The transfer function $\Gamma_b(z^T) = \mathbf{D}_b + \mathbf{C}_b(z^T I - \mathbf{A}_b)^{-1}\mathbf{B}_b$ of (24), see [2], [23], has the circulant-like structure as

$$\begin{pmatrix} H_0(z) & H_{T-1}(z) & \dots & H_2(z) & H_1(z) \\ zH_1(z) & H_0(z) & H_{T-1}(z) & \dots & H_2(z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ zH_{T-2}(z) & \dots & zH_1(z) & H_0(z) & H_{T-1}(z) \\ zH_{T-1}(z) & zH_{T-2}(z) & \dots & zH_1(z) & H_0(z) \end{pmatrix},$$

where $H_0(z) = D + \mathbf{C}(zI - \mathbf{A}^T)^{-1}\mathbf{A}^{T-1}\mathbf{B}$ and $H_k(z) = \mathbf{C}(zI - \mathbf{A}^T)^{-1}\mathbf{A}^{k-1}\mathbf{B}$, $k = 1,\dots,T-1$. It is worthwhile mentioning that the blocked transfer function has the structure of a generalized circulant matrix. The theory of generalized circulant matrices is very similar to that of classical circulant matrices; see [9]. Using such techniques we obtain the following result.

In order to deal with the zeros of the system (25), we first need to review the following result from [41], obtained by specializing Lemma 1 of [41] to the time-invariant case.

Lemma 3 *[41] Let* $\widetilde{\mathbf{A}}_b = I_T \otimes \mathbf{A}$, $\widetilde{\mathbf{B}}_b = I_T \otimes \mathbf{B}$, $\widetilde{\mathbf{C}}_b = I_T \otimes \mathbf{C}$ *and* $\widetilde{\mathbf{D}}_b = I_T \otimes \mathbf{D}$. *Furthermore, define* $\mathbf{E}_\zeta \triangleq \begin{pmatrix} 0 & 1 & & 0 \\ 0 & & \ddots & \\ \vdots & & \ddots & 1 \\ \zeta & 0 & & 0 \end{pmatrix}$, $\mathbf{E}_\zeta \in \mathbb{C}^{T\times T}$ *and* $\widetilde{\mathbf{E}}_\zeta = \mathbf{E}_\zeta \otimes I_{\bar{n}}$ *where* $\otimes$

denotes the Kronecker product and ζ denotes a complex number. Then there exist invertible matrices T_l and T_r and matrices X and Y such that for all $\zeta \in \mathbb{C}$

$$\begin{pmatrix} I_{\bar{n}(T-1)} & 0 & 0 \\ 0 & \zeta I - \mathbf{A}_b & -\mathbf{B}_b \\ 0 & \mathbf{C}_b & \mathbf{D}_b \end{pmatrix} = \begin{pmatrix} T_l & 0 \\ X & I \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{E}}_\zeta - \widetilde{\mathbf{A}}_b & -\widetilde{\mathbf{B}}_b \\ \widetilde{\mathbf{C}}_b & \widetilde{\mathbf{D}}_b \end{pmatrix} \begin{pmatrix} T_r & Y \\ 0 & I \end{pmatrix}. \tag{27}$$

Using this lemma we introduce the following result.

Proposition 1 *Let Φ denote the Fourier matrix of the proper dimension and $\Gamma(z) = Q(z)^{-1}P(z)$ be a left coprime factorization of the network transfer function. Consider the system matrices*

$$\Sigma_b(z) = \begin{pmatrix} zI_{\overline{n}} - \mathbf{A}_b & -\mathbf{B}_b \\ \mathbf{C}_b & \mathbf{D}_b \end{pmatrix},$$

$$\hat{\Sigma}_b(z) = \begin{pmatrix} I_{\overline{n}(T-1)} & 0 \\ 0 & \Sigma_b(z) \end{pmatrix}.$$

There exist invertible matrices $L(z)$ and $R(z)$ that are invertible for all nonzero complex numbers $z \in \mathbb{C}$ such that

$$\hat{\Sigma}_b(z^T) = L(z) \begin{pmatrix} \begin{pmatrix} zI_{\overline{n}} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & D \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \omega^{T-1}zI_{\overline{n}} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & D \end{pmatrix} \end{pmatrix} R(z). \tag{28}$$

Proof. First, observe that the following equality holds

$$\mathbf{E}_1 = \Phi \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{T-1} \end{pmatrix} \Phi^*, \tag{29}$$

where Φ is the Fourier matrix of the proper dimension. Furthermore, we have

$$\mathbf{E}_\zeta = z\Delta(z)\mathbf{E}_1\Delta(z)^{-1}. \tag{30}$$

where $\Delta(z) = \begin{pmatrix} 1 & & & \\ & z & & \\ & & \ddots & \\ & & & z^{T-1} \end{pmatrix}$.

Now by using (30) and (29), one can easily verify that the following equality holds

$$\widetilde{\mathbf{E}}_\zeta = (\Delta(z)\Phi) \otimes I_{\bar{n}} \begin{pmatrix} zI_{\bar{n}} & & & \\ & \omega z I_{\bar{n}} & & \\ & & \ddots & \\ & & & \omega^{T-1}zI_{\bar{n}} \end{pmatrix} (\Delta(z)\Phi)^{-1} \otimes I_{\bar{n}}. \tag{31}$$

Therefore, for any $\zeta \neq 0$, $z^T = \zeta$, we have $\tilde{T}(z) \triangleq \Delta(z)\Phi \otimes I_{\bar{n}}$, $\tilde{R}(z) \triangleq \Delta(z)\Phi \otimes I_m$ and $\tilde{L}(z) \triangleq \Delta(z)\Phi \otimes I_p$. Hence,

$$\begin{pmatrix} \widetilde{\mathbf{E}}_\zeta - \widetilde{\mathbf{A}}_b & -\widetilde{\mathbf{B}}_b \\ \widetilde{\mathbf{C}}_b & \widetilde{\mathbf{D}}_b \end{pmatrix} = \left(\begin{array}{c|c} \tilde{T}(z) & 0 \\ \hline 0 & \tilde{L}(z) \end{array}\right) \left(\begin{array}{c|c} \begin{pmatrix} zI - A & & 0 \\ & \ddots & \\ 0 & & \omega^{T-1}zI - A \end{pmatrix} & \begin{pmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{pmatrix} \\ \hline \begin{pmatrix} C & & 0 \\ & \ddots & \\ 0 & & C \end{pmatrix} & \begin{pmatrix} D & & 0 \\ & \ddots & \\ 0 & & D \end{pmatrix} \end{array}\right) \left(\begin{array}{c|c} \tilde{T}^{-1}(z) & 0 \\ \hline 0 & \tilde{R}^{-1}(z) \end{array}\right). \tag{32}$$

Now by substituting (32) into the equation (27) and performing the required rows and columns reordering, the conclusion of the proposition becomes immediate.

□

The preceding results imply the following characterization of the finite zeros for the interconnected systems. Thus consider the interconnected system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ defined in (5). Let $(\mathbf{A_b}, \mathbf{B_b}, \mathbf{C_b}, D_b)$ denote the associated blocked system, defined as in (25) and (26).

Theorem 7 *A complex number $\zeta_0 \neq 0$ is a finite zero of the blocked network $(\mathbf{A_b}, \mathbf{B_b}, \mathbf{C_b}, D_b)$ if and only if there exists $z_0 \in \mathbb{C}$ with $z_0^T = \zeta_0$ such that z_0 is a finite zero of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$.*

Proof. Necessity. Suppose that $\zeta_0 = z_0^T$ is a zero of the system matrix $\Sigma_b(\zeta)$, then by recalling the result of Proposition 1, one can easily see that one or more of diagonal blocks in (28) should have rank below their normal rank i.e. there exist at least one T-th root of ζ_0 which is a zero of the unblocked system.
Sufficiency. Suppose that z_0 is a zero of the unblocked system (24). Then at least one of the diagonal blocks in (28) loses rank below its normal rank. Now, again by using (28), one can conclude that $\zeta_0 = z_0^T$ is a zero of $\hat{\Sigma}_b(z)$. The latter implies that ζ_0 must be a zero of the system (25).

□

The above theorem only treats the finite nonzero zeros. To treat the other cases i.e. zeros at the origin and infinity, we recall the following result from [42].

Proposition 2 *Consider the unblocked networked system (24) with transfer function $\Gamma(z)$ and the blocked networked system (25) with transfer function $\Gamma_b(\zeta)$. Suppose that the quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ is minimal. Then*

(1) The system (24) has a zero at $z = \infty$ if and only if the system (25) has a zero at $\zeta = \infty$.
(2) The system (24) has a zero at the origin if and only if the the system (25) has a zero at the origin.

This implies the next characterization of the zeros for the systems (24) and (25).

Theorem 8 *Let D be full rank and $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ a homogeneous network with SISO agents. Then the blocked network $(\mathbf{A_b}, \mathbf{B_b}, \mathbf{C_b}, D_b)$ has no zeros at infinity. The finite zeros of $(\mathbf{A_b}, \mathbf{B_b}, \mathbf{C_b}, D_b)$ are exactly all $\zeta = z^T$ such that $h(\omega^k z)$ is a finite zero of (L, R, S, D) for some $0 \leq k \leq T-1$.*

Proof. The proof readily follows from Proposition 2 and the first part of Theorem 4. □

5 Conclusions

In this paper, we explored the zeros of networks of linear systems. It was assumed that the interaction topology is time-invariant. The zeros were characterized for both homogeneous and heterogeneous networks. In particular, it was shown that for homogeneous networks with full rank direct feedthrough matrix, the finite zeros of the whole network are exactly the preimages of interconnection dynamics zeros under the inverse of an agent transfer function. We then discussed the condition under which the networked systems have no finite nonzero zeros. Then *generalized circulant matrices* were used for a concise analysis of the finite nonzero zeros of blocked networked systems. Moreover, we recalled some results about their zeros at infinity and at the origin. It was shown that the networked systems have zeros at the origin (infinity) if and only if their associated blocked systems have zeros at the origin (infinity). As a part of our future work we will address open problems such as the consideration of periodically varying network topologies and MIMO dynamics for each agents. Furthermore, as explained in the illustrative example given in the current paper, adding and removing links can dramatically change the zero structure. Thus, another interesting research direction involves exploring how links in the networked systems can be systematically designed such that the resultant networked systems attain a particular zero dynamics. Moreover, the issues involved with robustness property of zero-dynamics attacks remain still open. We also believe that zero-dynamics attacks require further investigations especially when there exist some noise components involved with the input signal. It is also an interesting research question to study how zeros positions with respect to a unit circle affect an attack policy.

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